Skew-Prime Polynomial Matrices: The Polynomial-Model Approach*

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ABSTRACT

We examine the concept of skew-primeness of polynomial matrices in terms of the associated polynomial model. It is shown that skew-primeness can be characterized in terms of the property of decomposition of a vector space relative to an endomorphism. This basic result is then applied to the special case of nonsingular polynomial matrices. We investigate the nonuniqueness of skew-complements of a skew-prime pair. It is shown that the space of equivalence classes of skew-complements is in bijective correspondence with a finite-dimensional linear space. Finally, the equivalence of the solutions to the problem of output regulation with internal stability obtained via geometric methods and via polynomial matrix techniques is shown explicitly.

1. INTRODUCTION

An impressive amount of literature has grown in the last few years to show that problems of linear algebra and linear system theory can be fruitfully studied using polynomial methods. In the context of modern algebraic system theory, polynomial modules played a critical role in the completely satisfactory resolution of the realization problem by Kalman [15]. More recently, Fuhrmann [9, 10] described a new approach to the study of finite-dimensional linear systems using polynomial models. This theory of polynomial models is

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by far the best tool to reconcile and unify some seemingly unrelated approaches to linear system theory: the geometric approach using state-space concepts developed by Basile and Marro [2] and Wonham [31] and the approach based on polynomial fractional representations developed by Rosenbrock [24] and Wolovich [28].

The basic idea of Fuhrmann [9, 10] is to associate a state-space model with a polynomial matrix fractional representation of the transfer matrix. This state-space model relates the fractional representations to the abstract module-theoretic framework. This polynomial model has proved to be very useful in relating and clarifying seemingly unrelated concepts. For example, Fuhrmann [9] showed that the concept of coprimeness of polynomial matrices is intimately related with the basic system-theoretic concepts of reachability and observability. Several other system-theoretic problems were studied using this polynomial model: strict system equivalence (by Fuhrmann [10]), linear feedback (by Fuhrmann [11]; also see Hautus and Heymann [14]), etc.

Emre and Hautus [8] were the first to use this polynomial model to study (F, G)-invariant subspaces in terms of polynomial matrices. Antoulas [1], Fuhrmann and Willems [12], and Fuhrmann [11] obtained somewhat different results on (F, G)-invariant subspaces. Several further results on the connection between the concepts of geometric control theory and polynomial matrix methods were obtained by Khargonekar and Emre [19], essentially following the setup of Emre and Hautus [8]. This has set the stage for a systematic investigation of solutions to linear-control problems in terms of the polynomial model.

Recently, the concept of *skew-primeness* of polynomial matrices has arisen in the resolution of certain linear multivariable control problems using polynomial matrix techniques. (See Wolovich [29] and Wolovich and Ferreira [30].) At first glance, the notion of skew-primeness seems to be a purely technical construct having no interpretation in the state-space or linear-algebraic terms. This paper is devoted to a systematic study of the concept of skew-primeness using the polynomial model of Fuhrmann. Our results show that *skew-primeness of polynomial matrices is closely related to decomposition of a vector space relative to an endomorphism.* This ties up neatly with the existing results on factorizations of a polynomial matrix. It is known that F-invariant subspaces of linear space with a linear map F can be characterized in terms of factorizations of a polynomial matrix. (See Antoulas [1], Emre [7], and Fuhrmann and Willems [12].) Our results show that F-invariant subspaces which have complementary F-invariant subspaces can be characterized in terms of skew-primeness of the factors of the polynomial matrix. (See Theorems 3.3, 3.12, and 4.3.)

Thus the concept of skew-primeness has a completely satisfactory linear-algebraic interpretation. Therefore it is not at all surprising that our abstract results have a system-theoretic application. We show that the solutions to the problem of output regulation with internal stability obtained through geometric methods and through polynomial matrix methods are in fact equivalent. This result reveals some of the deep connections that exist between the geometric approach and the polynomial matrix techniques.

It is well known that for a skew-prime pair, the *skew-complement* may not be unique. (See Section 4 for various definitions.) Wolovich [29] showed that under some conditions, the skew-complement is unique up to the action of unimodular matrices. Our results show that for a skew-prime pair of nonsingular polynomial matrices, *the space of (equivalence classes of) skew-complements is in bijective correspondence with a finite-dimensional linear space.* The dimension of this linear space can be computed easily from the elementary divisors of the polynomial matrices involved. As a simple corollary of our result, we show that the sufficient conditions of Wolovich [29, Theorem 6] for the uniqueness of the skew-complement are necessary as well.

The paper is organized as follows: In Section 2 we review the techniques and results of Fuhrmann [9], Emre and Hautus [8], and Khargonekar and Emre [19]. Section 3 is devoted to a study of skew-primeness using the polynomial model. Here we characterize skew-primeness in terms of decomposition of a polynomial module. We study the special case of nonsingular matrices in Section 4. Finally, the results of Section 3 are applied to the problem of output regulation with internal stability in Section 5. We explicitly show that the conditions for solvability of the problem of output regulation with internal stability obtained by geometric methods and by polynomial matrix methods are equivalent.

2. POLYNOMIAL MODELS FOR LINEAR SYSTEMS

In this section we will give a brief description of the techniques and results developed by Fuhrmann [9, 10], Emre and Hautus [8], and Khargonekar and Emre [19].

Let K be an arbitrary field. (In Section 5 we will assume that $K=\mathbb{R}$, the field of real numbers.) Let V be a K-vector space. We denote by $V[z]$ the K-vector space of polynomials in the indeterminate in z with coefficients in V. Clearly, $V[z]$ admits a natural $K[z]$-module structure. Let $V((z^{-1}))$ denote the K-vector spaces of all formal Laurent series in z^{-1} with coefficients in V. Again $V((z^{-1}))$ admits a natural $K((z^{-1}))$-vector space structure. Let $K(z)$ denote the quotient field of $K[z]$. Note that any element p/q in $K(z)$ may be uniquely identified with a Laurent series in z^{-1} (obtained, for example, by long division of p by q). For any element x in $V((z^{-1}))$, let $(x)_+$, $(x)_-$, and

$(x)_{-n}$ respectively denote the polynomial part of x, the strictly proper part of x, and the coefficient of z^{-n}. Any x in $V((z^{-1}))$ is called *strictly proper* iff $(x)_- = x$, and is called *proper* iff $z^{-1}x$ is strictly proper.

A *finite-dimensional, linear, time-invariant system* Σ (over K) is a quadruple (F, G, H, X) where X is a finite-dimensional K-linear space and $F: X \to X$, $G: K^m \to X$, $H: X \to K^p$ are K-linear maps. (As our results are purely algebraic, they apply to continuous-time as well as discrete-time systems.) A subspace $V \subseteq X$ is called *(F, G)-invariant* iff

(2.1) $$FV \subseteq V + \operatorname{im} G.$$

The set of all (F, G)-invariant subspaces in $\ker H$ is closed under addition and has a largest element, which is denoted by V_m. If $V \subseteq X$ is an (F, G)-invariant subspace, then the set

$$\mathbf{L}(V) := \{\text{linear map } L: X \to K^m : (F + GL)V \subseteq V\}$$

is nonempty. An (F, G)-invariant subspace V is called a *reachability subspace* iff V is the smallest $(F + GL)$-invariant subspace containing $V \cap \operatorname{im} G$, for any (or some) L in $\mathbf{L}(V)$. The set of all reachability subspaces in $\ker H$ is also closed under addition and has a maximal element, which is denoted by R_m. The concepts of (F, G)-invariant and reachability subspaces are the fundamental concepts of the so-called geometric control theory. For a detailed development of these concepts and their applications to linear control problems, the reader is referred to Basile and Marro [2] and Wonham [31].

Let V be a K-linear space and $F: V \to V$ be a linear map. It is well known (see, e.g., [16, Chapter 10]) that F induces a $K[z]$-module structure on V; for any $p(z)$ in $K[z]$, let $p(z)\cdot x := p(F)x$ for all x in V. If V is finite-dimensional over K, then V becomes a torsion $K[z]$-module. Conversely, given a $K[z]$-module V, we can define a K-linear map $F: V \to V$; for any x in V, let $Fx = z \cdot x$. Thus, there is a one-to-one correspondence between $K[z]$-modules and K-vector spaces with an endomorphism.

In the polynomial matrix approach to the study of linear systems, the transfer matrix of the system is usually represented in the form

(2.2) $$Z = PQ^{-1}R + U$$

where P, Q, R, U are polynomial matrices of appropriate sizes. Recall that a linear system $\Sigma = (F, G, H, X)$ is called a *realization* of the (strictly proper) transfer matrix Z iff

$$Z=\sum_{t=1}^{\infty} HF^{t-1}Gz^{-t}=H(zI-F)^{-1}G.$$

It is clear that in order to relate the polynomial matrix theory to the state-space or the geometric theory, it is necessary to associate a state-space model with the polynomial matrix fraction representation (2.2) of the transfer matrix. This key step was taken by Fuhrmann [9, 10].

Let T be a $p\times m$ polynomial matrix. Then

$$X_T:=\{x \text{ in } K^p[z] : \text{there exists a strictly proper } y \text{ such that } x=Ty\}.$$

In particular, if T is square and nonsingular, then

$$X_T:=\{x \text{ in } K^p[z] : T^{-1}x \text{ is strictly proper}\}.$$

In this case we define the projection map

$$\pi_T: K^p[z]\to X_T: x\to T(T^{-1}x)_-.$$

Given the fractional representation (2.2), let us define the K-linear maps

$$\begin{aligned} &F_Q: X_Q\to X_Q: x\to \pi_Q(zx),\\ &G_Q: K^m\to X_Q: x\to \pi_Q(Ru),\\ &H_Q: X_Q\to K^p: x\to (PQ^{-1}x)_{-1}. \end{aligned} \tag{2.3}$$

The following result associates a natural state-space realization with the fractional representation (2.2).

THEOREM 2.4 (Fuhrmann [9, 10]). *Let Z be a $p\times m$ strictly proper transfer matrix. Let P, Q, R, U be polynomial matrices such that Q is nonsingular and*

$$Z=PQ^{-1}R+U$$

Then $\Sigma(P,Q,R,U):=(F_Q,G_Q,H_Q,X_Q)$ is a realization of Z. Further,

$\Sigma(P, Q, R, U)$ is reachable if and only if Q and R are left coprime, and is observable if and only if P and Q are right coprime.

In case, $Z = Q^{-1}R$ (i.e., $P = I$, $U = 0$), we will use the notation $\Sigma(Q, R)$ instead of $\Sigma(I, Q, R, 0)$.

We will now recapitulate the results of Emre and Hautus [8] and Khargonekar and Emre [19] in a slightly modified form. Let R be a $p \times m$ polynomial matrix. Then X_R turns out to be a finite-dimensional K-linear space. Let Y be a strictly proper matrix such that the columns of $S := RY$ constitute a basis for X_R. For any x in X_R, there exists a unique constant vector g such that $x = Sg = RYg$. Then

$$z \cdot x := R(zYg)_- \tag{2.5}$$

defines a $K[z]$-module structure on X_R. We call this module structure a *shift-module structure* or *Y-shift-module structure*, whenever an explicit reference to Y is desired. The following result can be verified easily.

LEMMA 2.6. *Let R be a $p \times m$ polynomial matrix. Let Y be a strictly proper matrix such that the columns of $S := RY$ constitute a basis for the K-linear space X_R. Then there exist constant matrices H and F such that $S = RH(zI - F)^{-1}$ and the $H(zI - F)^{-1}$-shift-module structure is the same as the Y-shift-module structure.*

It is easy to see that F of the above lemma is unique. It is the matrix representation of the linear map $X_R \to X_R : x \mapsto z \cdot x$ with respect to the columns of S taken as a basis for X_R. On the other hand one choice for H is $(Y)_{-1}$. The shift-module structure was introduced by Khargonekar and Emre [19] for the particular form of Y as $H(zI - F)^{-1}$; the above lemma relates our setup to theirs.

In particular, if the polynomial matrix R is square and nonsingular, then all shift-module structures coincide, i.e., the shift-module structure is unique. It is easy to verify that in this case

$$z \cdot x = \pi_R(zx),$$

for all x in X_R. Thus in case of nonsingular polynomial matrices our shift-module structure coincides with the classical one due to Fuhrmann [9].

We can now describe the main results of Emre and Hautus [8], which give a closed-form expression for V_m of $\Sigma(Q, R)$.

THEOREM 2.7 (Emre and Hautus [8, Section 3]). *Let $Z = Q^{-1}R$ be a strictly proper transfer matrix in a left matrix fraction representation. Let $(F, G, H, X_Q) := \Sigma(Q, R)$. Then X_R is the largest (F, G)-invariant subspace in* $\ker H$. *Further, for any shift-module structure on X_R, there exists an L in $\mathbf{L}(X_R)$ such that the $K[z]$-module structure induced by $F + GL$ is the same as the given shift-module structure. Conversely, for any L in $\mathbf{L}(X_R)$, there exists a strictly proper matrix Y such that the $K[z]$-module structure on X_R induced by $F + GL$ is the same as the Y-shift-module structure on X_R.*

A similar closed-form expression for V_m of $\Sigma(P, Q, R, U)$ is fully investigated by Emre and Hautus [8] and Khargonekar and Emre [19].

We will also need the following result in the following sections.

THEOREM 2.8 (Khargonekar and Emre [19, Corollary (5.6)]). *Let $Z = Q^{-1}R$ be a strictly proper transfer matrix in the left matrix fraction representation. Let $(F, G, H, X_Q) := \Sigma(Q, R)$. Let $V_m = X_R$ and R_m respectively denote the largest (F, G)-invariant and reachability subspaces in* $\ker H$. *Then for any L in $\mathbf{L}(V_m)$, the nontrivial invariant factors of the linear map induced by $F + GL$ on V_m / R_m are the same as the nontrivial invariant factors of the polynomial matrix R.*

We will now describe some auxiliary results of independent interest, some of which will also be useful in the subsequent developments.

It is well known that factorizations of polynomial matrices are closely related with (F, G)-invariant subspaces. See Antoulas [1], Emre [7], Fuhrmann and Willems [12], Fuhrmann [11], and Khargonekar and Emre [19] for some results illustrating this relation. The following proposition is another such result.

PROPOSITION 2.9. *Let R be a $p \times m$ polynomial matrix. Let D be a $p \times p$ nonsingular and E be a $p \times m$ polynomial matrix such that $R = DE$. Then DX_E is a submodule of X_R for any shift-module structure on X_R.*

Proof. Let Y be a strictly proper matrix such that the columns of $S := RY$ constitute a basis for X_R, and consider the Y-shift-module structure on X_R. Suppose x belongs to DX_E. Then there exists a unique constant vector g such that $x = Sg$. Then

$$z \cdot x = R(zYg)_- = zRYg - RHg,$$

where $H := (Y)_{-1}$. Therefore

$$D^{-1}(z \cdot x) = zEYg - EHg.$$

Since x belongs to DX_E, it follows that EYg is polynomial. Hence $D^{-1}(z \cdot x)$ is also polynomial. Thus, $z \cdot x$ belongs to DX_E for all x in DX_E. ■

REMARK. The converse of the above proposition is also true. In other words, *if $M \subseteq X_R$ is a (nonzero) submodule of X_R for any shift-module structure on X_R, then there exist polynomial matrices D, E such that D is nonsingular*, $R = DE$, and $M = DX_E$. These ideas are closely related to the largest reachability space. Several results in this direction have been obtained by Fuhrmann [11] in a slightly different form. In fact, an independent proof of Proposition 2.9 may be given using Corollary (4.8) of [11]. On the other hand, Corollary (4.8) of [11] can be strengthened using the converse of Proposition 2.9 stated above. We will not explore these ideas in detail here, as they are not directly related to the subject of this paper.

3. A GEOMETRIC INTERPRETATION OF SKEW-PRIMENESS

Let D be a $p \times p$ nonsingular polynomial matrix and R be a $p \times m$ polynomial matrix. Recall that the ordered pair (D, R) is called *skew-prime* iff there exist $\tilde{R}$ and $\tilde{D}$ in $K[z]^{p \times m}$ and $K[z]^{m \times m}$ respectively such that

(3.1i) $DR = \tilde{R}\tilde{D}$,
(3.1ii) D and $\tilde{R}$ are left coprime, and
(3.1iii) R and $\tilde{D}$ are right coprime.

The pair $(\tilde{R}, \tilde{D})$ is called a *skew-complement* of (D, R). Intuitively, skew-primeness is a combination of commutativity and coprimeness. The concept of skew-primeness plays an important role in the polynomial matrix approach to certain control-theoretic problems such as output regulation with internal stability (see Wolovich and Ferreira [30]) and stochastic control (see Kučera [20]).

In this section we give a geometric characterization of the concept of skew-primeness. In particular, we show that the pair (D, R) is skew prime if and only if the submodule DX_R of the module X_{DR} is a $K[z]$-direct summand and $\dim X_{DR}$ equals the sum of $\dim X_D$ and $\dim X_R$. This characterization leads to two significant applications, which are considered in the next two sections. Under the assumption that R is square and nonsingular, we show that the set of skew-complement pairs $(\tilde{R}, \tilde{D})$ modulo unimodular equivalence can be parametrized by an $\mathbb{R}$-linear space. In Section 5, we apply this geometric characterization to prove that the conditions for the solvability of the problem of output regulation with internal stability obtained via the geometric methods and polynomial matrix methods are equivalent.

We start by listing some alternative characterizations of the concept of skew-primeness.

LEMMA 3.2. *Let D be a $p \times p$ nonsingular polynomial matrix and R be a $p \times m$ polynomial matrix. Then the following statements are equivalent:*

(i) *The ordered pair (D, R) is skew-prime.*

(ii) *There exist polynomial matrices A, B such that $AD + RB = I$.*

(iii) *There exist polynomial matrices $\tilde{R}, \tilde{D}$ such that $DR = \tilde{R}\tilde{D}$, $\det D = \det \tilde{D}$, and $D, \tilde{R}$ are left coprime. (In this case $(\tilde{R}, \tilde{D})$ is a skew-complement of (D, R).)*

(iv) *There exist polynomial matrices $\tilde{R}, \tilde{D}$ such that $DR = \tilde{R}\tilde{D}$, $\det D = \det \tilde{D}$, and $R, \tilde{D}$ are right coprime. (In this case, $(\tilde{R}, \tilde{D})$ is a skew-complement of (D, R).)*

For a proof of the above result, see Wolovich [29].

We will now prove the main result of this section, relating the concept of skew-primeness with the properties of the associated linear spaces X_{DR} and DX_R.

THEOREM 3.3. *Let D be a $p \times p$ nonsingular polynomial matrix and R be a $p \times m$ polynomial matrix. Then the ordered pair (D, R) is skew-prime if and only if the following conditions hold:*

$$\dim X_{DR} = \dim X_D + \dim X_R, \tag{3.4}$$

and there exists $M \subseteq X_{DR}$ such that

$$X_{DR} = DX_R \underset{K[z]}{\oplus} M \tag{3.5}$$

for some shift-module structure on X_{DR}.

Proof. We will first prove the "if" part. Suppose there exists a strictly proper matrix Y such that the columns of $S := DRY$ constitute a basis for X_{DR} and there exists $M \subseteq X_{DR}$ such that

$$X_{DR} = DX_R \underset{K[z]}{\oplus} M, \tag{3.5}$$

for the Y-shift-module structure. Let G be a constant matrix such that the columns of $S_M: = SG = DRYG$ constitute a basis for M. Since M is a $K[z]$-submodule of X_{DR}, there exists a constant matrix F such that

$$z \cdot S_M = S_M F,$$

where $z \cdot S_M$ denotes the matrix resulting after the action of z on the columns of S_M. By definition,

$$z \cdot S_M = DR(zYG)_- = zDRYG - DRH = zS_M - DRH$$

for some constant matrix H. Therefore $S_M = DRH(zI - F)^{-1}$. It is easy to see that (H, F) is observable, since the columns of S_M constitute a basis for M. Let $\tilde{D}, \tilde{S}$ be left coprime polynomial matrices such that

$$H(zI - F)^{-1} = \tilde{D}^{-1}\tilde{S}.$$

It now follows (from the various coprimeness conditions) that

$$\deg\det \tilde{D} = \deg\det(zI - F) = \dim M. \tag{3.6}$$

We also have

$$D^{-1}S_M = R\tilde{D}^{-1}\tilde{S}. \tag{3.7}$$

We will now prove that $D^{-1}S_M$ and $R\tilde{D}^{-1}\tilde{S}$ are coprime factorizations. Consider the K-linear map

$$\varphi : X_{DR} \to X_D : x \to \pi_D(x).$$

Clearly, $DX_R \subseteq \ker\varphi$. Conversely, any x in X_{DR} belongs to $\ker\varphi$ only if $D^{-1}x$ is polynomial. Hence, $\ker\varphi = DX_R$. As D is nonsingular, it follows that

$$\dim(\ker\varphi) = \dim X_R = \dim X_{DR} - \dim X_D.$$

Consequently, φ is surjective. Further, (3.5) implies that $\varphi \mid M$ is also surjective. Hence, there exists a constant matrix B such that

$$\varphi(S_M B) = \pi_D(S_M B) = \pi_D(I).$$

Therefore, there exists a polynomial matrix A such that

$$DA + S_M B = I.$$

Thus, D and S_M are left coprime. We also have

$$\deg\det D = \dim X_D = \dim M = \deg\det \tilde{D}. \tag{3.8}$$

Now, as D and S_M are left coprime, (3.7) and (3.8) imply that R and $\tilde{D}$ must be right coprime.

Thus, $D^{-1}S_M$ and $R\tilde{D}^{-1}\tilde{S}$ are coprime factorizations of the same rational matrix. It is well known (see [24]) that

$$\det D = \det \tilde{D}.$$

Since $\tilde{D}$ and $\tilde{S}$ are left coprime, there exist polynomial matrices T_1 and T_2 such that

$$\tilde{D}T_1 + \tilde{S}T_2 = I.$$

Consequently,

$$DR\tilde{D}^{-1} = DRT_1 + DR\tilde{D}^{-1}\tilde{S}T_2.$$

Let $\tilde{R} := DRT_1 + S_M T_2$. We now have

$$DR = \tilde{R}\tilde{D}, \tag{3.9}$$

where $\tilde{D}$ and R are right coprime and $\det D = \det \tilde{D}$. By Lemma 3.2, (D, R) is skew-prime. This concludes the proof of the "if" part of the theorem.

We will now prove the "only if" part of the theorem. Suppose that (D, R) is skew-prime. Then there exist polynomial matrices $\tilde{R}, \tilde{D}$ such that (3.1) holds. We will first show that

$$X_{DR} = DX_R \underset{K}{\oplus} \tilde{R}X_{\tilde{D}}. \tag{3.10}$$

Clearly, $DX_R \subseteq X_{DR}$ and $\tilde{R}X_{\tilde{D}} \subseteq X_{\tilde{R}\tilde{D}} = X_{DR}$ are K-linear subspaces. Suppose x belongs to $DX_R \cap \tilde{R}X_{\tilde{D}}$. Then $x = \tilde{R}x_1$ for some x_1 in $X_{\tilde{D}}$. As x also belongs to DX_R,

$$D^{-1}x = D^{-1}\tilde{R}x_1 = R\tilde{D}^{-1}x_1$$

is a polynomial vector. Since R and $\tilde{D}$ are right coprime, there exist polynomial matrices A_1 and A_2 such that $A_1R + A_2\tilde{D} = I$. Consequently,

$$\tilde{D}^{-1}x_1 = A_1R\tilde{D}^{-1}x_1 + A_2x_1.$$

Since $R\tilde{D}^{-1}x_1$ is polynomial, $\tilde{D}^{-1}x_1$ is also polynomial. But x_1 belongs to $X_{\tilde{D}}$; hence $x_1 = 0$. Thus $x = 0$. We can now conclude that $DX_R \cap \tilde{R}X_{\tilde{D}} = \{0\}$.

In order to complete the proof of (3.10), we will now show that $X_{DR} \subseteq DX_R + \tilde{R}X_{\tilde{D}}$. Let x be in X_{DR}. As D and $\tilde{R}$ are left coprime, there exist polynomial matrices B_1 and B_2 such that $DB_1 + \tilde{R}B_2 = I$. Let $x_1 := B_1x$, $x_2 := B_2x$. Then

$$x = Dx_1 + \tilde{R}x_2.$$

Let $x_3 := \pi_{\tilde{D}}(x_2)$ and

$$x_4 := x_1 + R\tilde{D}^{-1}(x_2 - x_3).$$

Note that $\tilde{D}^{-1}(x_2 - x_3)$ is polynomial and hence x_4 is also polynomial. Now

$$\begin{aligned} x = Dx_1 + \tilde{R}x_2 &= Dx_1 + \tilde{R}\tilde{D}\left[\tilde{D}^{-1}(x_2 - x_3)\right] + \tilde{R}x_3 \\ &= D\left[x_1 + R\tilde{D}^{-1}(x_2 - x_3)\right] + \tilde{R}x_3 \\ &= Dx_4 + \tilde{R}x_3. \end{aligned}$$

We will now show that x_4 belongs to X_R. Since x belongs to X_{DR}, there exists a strictly proper vector y such that $x = DRy$. Then

$$Dx_4 = DRy - DR\tilde{D}^{-1}x_3$$

or, equivalently,

$$x_4 = R\left(y - \tilde{D}^{-1}x_3\right).$$

Since x_3 belongs to $X_{\tilde{D}}$, it follows that $\tilde{D}^{-1}x_3$ is strictly proper. Thus x_4 belongs to X_R. We can now conclude that $X_{DR} \subseteq DX_R + \tilde{R}X_{\tilde{D}}$. Thus (3.10) holds.

We will now show that there exists a shift-module structure on X_{DR} such that (3.10) becomes a $K[z]$-direct-sum decomposition, and also that (3.4) holds. Consider the K-linear map

$$\theta: X_{\tilde{D}} \to \tilde{R}X_{\tilde{D}}: x \to \tilde{R}x.$$

Clearly, θ is surjective. Let x in $X_{\tilde{D}}$ be such that $\theta(x)=\tilde{R}x=0$. Then $DR\tilde{D}^{-1}x=0$, and hence $R\tilde{D}^{-1}x=0$. As R and $\tilde{D}$ are right coprime, there exist polynomial matrices A_1 and A_2 such that $A_1R+A_2\tilde{D}=I$. Consequently

$$\tilde{D}^{-1}x=A_2x.$$

As x is in $X_{\tilde{D}}$, $\tilde{D}^{-1}x$ is strictly proper. But A_2x is polynomial. Hence $x=0$. We conclude that θ is an isomorphism.

Since θ is an isomorphism, $\dim \tilde{R}X_{\tilde{D}}=\dim X_{\tilde{D}}$. But $\dim X_{\tilde{D}}=\dim X_D$, as $\det D$ is in fact equal to $\det \tilde{D}$ by Lemma 3.3. Thus, $\dim \tilde{R}X_{\tilde{D}}=\dim X_D$. Also, $\dim DX_R=\dim X_R$, as D is nonsingular. Thus (3.10) implies that

$$\dim X_{DR}=\dim X_D+\dim X_R.$$

Further, θ naturally induces a shift-module structure on $\tilde{R}X_{\tilde{D}}$. Let Y be a strictly proper matrix such that the columns of $\tilde{D}Y$ constitute a basis for $X_{\tilde{D}}$. As θ is an isomorphism, the columns of $\tilde{R}\tilde{D}Y$ constitute a basis for $\tilde{R}X_{\tilde{D}}$. Let $x=\tilde{R}\tilde{D}Yg$ be in $\tilde{R}X_{\tilde{D}}$. Then $z\cdot x:=\tilde{R}\tilde{D}(zYg)_-$ defines a $K[z]$-module structure on $\tilde{R}X_{\tilde{D}}$. [It is easy to see that $z\cdot x=\theta(\pi_{\tilde{D}}(z\theta^{-1}(x)))$.] Let $\hat{Y}$ be a strictly proper matrix such that the columns of $DR(Y{:}\hat{Y})$ form a basis for X_{DR}. Now consider the $(Y{:}\hat{Y})$-shift module structure of X_{DR}. By construction, $\tilde{R}X_{\tilde{D}}$ is a submodule of X_{DR}. By Proposition 2.9, DX_R is also a $K[z]$-submodule of X_{DR}. We conclude that

$$X_{DR}=DX_R \underset{K[z]}{\oplus} \tilde{R}X_{\tilde{D}}. \qquad \blacksquare$$

There is one drawback of the above theorem. The shift-module structure on X_{DR} is nonunique, and DX_R may be a $K[z]$-direct summand for some shift-module structure and fail to be a $K[z]$-direct summand for some other shift-module structure. Of course, this is very closely related to the problem of reachability spaces. We will now present another result which overcomes this drawback. (Also see Theorem 5.12 and Remark 5.17 in this connection.)

Let R be a $p\times m$ polynomial matrix of rank q [over $K(z)$]. Let $\epsilon_1,\epsilon_2,\dots,\epsilon_q$ be the invariant factors of R. (Note that 1's are also included in the invariant factors. Let N, P be unimodular polynomial matrices such that $R=N\Lambda P$, where Λ is the Smith canonical form of R. (See Newman [21] for the details of the Smith canonical form.) Let Δ be a $p \times p$ diagonal matrix defined as

$$\Delta := \operatorname{diag}(\epsilon_1, \epsilon_2, \dots, \epsilon_q, 0, 0, \dots, 0).$$

We now define

$$R_l := N\Delta.$$

Note that N, P above are not unique, in general. However, this nonuniqueness does not affect the following results. Our results remain valid for any choice of N and P. We first have the following preliminary

LEMMA 3.11. *Let D be a $p \times p$ nonsingular and R be a $p \times m$ polynomial matrix. Then (D, R) is skew-prime if and only if (D, R_l) is skew-prime.*

Proof. Let the rank (over $K(z)$) of R be q, and N, P, Λ, Δ be as above. Let J be a $p \times m$ constant matrix in the form

$$J = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

where I represents the $q \times q$ identity matrix. Then $R = R_l JP$.

Now suppose (D, R) is skew-prime. Then by Lemma 3.2 there exist polynomial matrices A, B such that $AD + RB = I$. Let $B_1 := JPB$. Then

$$AD + R_l B_1 = I.$$

Again by Lemma 3.2, (D, R_l) is skew-prime.

Conversely, suppose that (D, R_l) is skew-prime. Then by Lemma 3.2, there exist polynomial matrices X_1, X_2 such that

$$X_1 D + R_l X_2 = I.$$

Using the particular structure of J, it is easy to see that there exists a polynomial matrix X_3 such that $R_l X_2 = R_l J X_3$. Now letting $X_4 := P^{-1} X_3$, we have

$$X_1 D + R X_4 = I.$$

Hence, by Lemma 3.2, (D, R) is skew-prime. ■

The next result follows immediately from Theorem 3.3 and Lemma 3.11.

THEOREM 3.12. *Let D be a $p \times p$ nonsingular and R be a $p \times m$ polynomial matrix. Then the shift-module structure on X_{DR_l} is unique. Further, (D, R) is skew-prime if and only if there exists $M \subseteq X_{DR_l}$ such that*

$$X_{DR_l} = DX_{R_l} \underset{K[z]}{\oplus} M,$$

and

$$\dim X_{DR_l} = \dim X_D + \dim X_{R_l}.$$

Proof. Let q denote the rank of R. Let N, Δ, P be as above. Then $R_l = N\Delta$. Let Y_1 and Y_2 be strictly proper matrices such that the columns of $S = DR_lY_1 = DR_lY_2$ constitute a basis for X_{DR_l}. Let x be any element of X_{DR_l}. Then there exists a unique constant vector g such that $x = Sg = DR_lY_1g = DR_lY_2g$. It follows that $DR_l(Y_1 - Y_2)g = 0$, and consequently, $\Delta(Y_1 - Y_2)g = 0$. By the definition of Δ, we have

$$\Delta(Y_1 - Y_2)_{-j}g = 0. \tag{3.12}$$

for all integers j. Now for the Y_1-shift-module structure

$$z \cdot x = zDR_lY_1g - DR_l(Y_1)_{-1}g,$$

and for the Y_2-shift-module structure

$$z \cdot x = zDR_lY_2g - DR_l(Y_2)_{-1}g = zDR_lY_1g - DR_l(Y_1)_{-1}g,$$

where the last equality follows from (3.12). Thus, the Y_1-shift-module structure is exactly the same as the Y_2-shift-module structure. Hence the shift-module structure on X_{DR_l} is unique. The other half of the theorem follows immediately from Theorem 3.3 and Lemma 3.11. ■

REMARK 3.13. The uniqueness of the shift-module structure on X_{DR_l} should not come as any surprise. In fact, it is easy to see that the kernel of DR_l is generated by a constant matrix. It then follows from Corollary (5.8) of Emre and Hautus [8] that there is no nontrivial reachability subspace in X_{DR_l}. This explains the uniqueness of the shift-module structure on X_{DR_l}. Note that

R_l is not necessarily nonsingular. In fact, R_l is nonsingular if and only if the rank [over $K(z)$] of R is p. By (obvious) further matrix manipulations, one can reduce the problem to skew-primeness of nonsingular matrices.

4. THE NONSINGULAR CASE

We will now apply the results of the previous section to the special case of nonsingular polynomial matrices. So, let D and R be $p \times p$ nonsingular matrices. If (D, R) is skew-prime, then there exist $p \times p$ polynomial matrices $\tilde{R}, \tilde{D}$ such that (3.1) holds. Let us define

$$\mathrm{SC}(D, R) := \{(\tilde{R}, \tilde{D}) : (\tilde{R}, \tilde{D}) \text{ is a skew-complement of } (D, R)\}.$$

Thus, the elements of $\mathrm{SC}(D, R)$ represent matrices $(\tilde{R}, \tilde{D})$ that satisfy (3.1). Define an equivalence relation $\sim$ on $\mathrm{SC}(D, R)$ as follows: $(\tilde{R}_1, \tilde{D}_1) \sim (\tilde{R}_2, \tilde{D}_2)$ iff there exists a unimodular matrix M such that $\tilde{R}_2 = \tilde{R}_1 M$ and $\tilde{D}_1 = M\tilde{D}_2$. Let $[\tilde{R}, \tilde{D}]$ denote the equivalence class of $(\tilde{R}, \tilde{D})$ in $\mathrm{SC}(D, R)$. We denote the set of all such equivalence classes of $\mathrm{SC}(D, R)$ by $S(D, R)$. The main result of this section shows that the set $S(D, R)$ can be parametrized by a K-linear space whose dimension can be calculated from the elementary divisors of D and R. As a simple corollary of this result, we show that if $\det D$ and $\det R$ are relatively prime, then there is a unique (up to unimodular matrices) skew-complement to (D, R). (This last result has been previously obtained by Wolovich [29].) Thus, our results of this section constitute a significant generalization of certain results of Wolovich [29] and lead to a deeper understanding of the concept of skew-primeness.

Let $Q := DR$. Since Q is nonsingular, it follows from the results of Section 2 that there is a unique shift-module structure on X_Q given by

$$z \cdot x := \pi_Q(zx). \tag{4.1}$$

Note that X_Q can alternatively be viewed as a K-linear space with the endomorphism $F_Q: X_Q \to X_Q: x \mapsto z \cdot x$. From this point of view $K[z]$-submodules of X_Q are exactly the same as F_Q-invariant subspaces of X_Q. This point of view will be particularly useful in this section.

It is known that $K[z]$-submodules (or equivalently, F_Q-invariant subspaces) of X_Q are closely related to factorizations of the matrix Q. We summarize (some of) the previously known results in the following

LEMMA 4.2. *Let Q be a nonsingular polynomial matrix. If D and R are nonsingular polynomial matrices such that $Q = DR$, then DX_R is an F_Q-invariant subspace of X_Q. Conversely, if V is any F_Q-invariant subspace of X_Q, then there exist nonsingular polynomial matrices $\hat{D}$ and $\hat{R}$ such that $Q = \hat{D}\hat{R}$ and $V = \hat{D}X_{\hat{R}}$. Finally, let D_1, R_1, D_2, R_2 be nonsingular polynomial matrices such that $Q = D_1R_1 = D_2R_2$. Then $D_1X_{R_1} = D_2X_{R_2}$ if and only if there exists a unimodular polynomial matrix M such that $D_1 = D_2M$ and $R_2 = MR_1$.*

Thus F_Q-invariant subspaces of X_Q can be represented in the form DX_R where $Q = DR$. When does DX_R have an F_Q-invariant complement? This question is answered in the following

THEOREM 4.3. *Let D and R be $p \times p$ square nonsingular polynomial matrices, and let $Q := DR$. Then DX_R has an F_Q-invariant complement in X_Q if and only if the ordered pair (D, R) is skew-prime.*

Proof. As D and R are nonsingular,

$$\dim X_Q = \deg\det DR = \dim X_D + \dim X_R.$$

Now the result follows directly from Theorem 3.3. ■

The results stated in Lemma 4.2 may be found in Fuhrmann and Willems [12] and in Antoulas [1]. Theorem 4.3 can be considered as a reformulation of results presented in Fuhrmann and Willems [12, Theorem 2.13].

Let (D, R) be a given skew-prime ordered pair of nonsingular matrices. Then there exist nonsingular matrices $\tilde{R}$, $\tilde{D}$ such that (3.1) holds. We will now investigate some questions related to the uniqueness and parametrization (in case of nonuniqueness) of such skew-complement pairs. Let us define

$$\text{(4.4)} \qquad \Gamma(D, R) := \{M \subseteq X_{DR} : F_{DR}M \subseteq M \text{ and } X_{DR} = DX_R \oplus M\}.$$

Thus D and R are skew-prime if and only if $\Gamma(D, R)$ is nonempty. In fact, we have the following

PROPOSITION 4.5. *Let (D, R) be a skew-prime ordered pair of nonsingular polynomial matrices. Then there exists a bijective correspondence between the sets $S(D, R)$ and $\Gamma(D, R)$.*

Proof. Consider the map

$$\psi: S(D,R) \to \Gamma(D,R): [\tilde{R}, \tilde{D}] \to \tilde{R}X_{\tilde{D}}.$$

It follows from the proof of Theorem 3.3, Lemma 4.2, and the definitions of $S(D,R)$ and $\Gamma(D,R)$ that ψ is well defined and one-to-one. To prove that ψ is surjective, let us consider an F_{DR}-invariant complement in X_{DR} of DX_R. By Lemma 4.2 there exist nonsingular polynomial matrices $\tilde{R}, \tilde{D}$ such that $M = \tilde{R}X_{\tilde{D}}$ and $DR = \tilde{R}\tilde{D}$. Then proceeding as in the proof of the first part of Theorem 3.3, it is easy to show that $DX_R \cap \tilde{R}X_{\tilde{D}} = \{0\}$ implies that $\tilde{D}$ and R are right coprime and $X_{DR} \subseteq DX_R + \tilde{R}X_{\tilde{D}}$ implies that D and $\tilde{R}$ are left coprime. Hence, $(\tilde{R}, \tilde{D})$ belongs to $SC(D,R)$ and $\psi([\tilde{R}, \tilde{D}]) = M$. Thus, ψ is a bijection. ∎

Thus, we see that there is a bijective correspondence between the set of skew-complements (modulo unimodular equivalence) of (D,R) and F_{DR}-invariant complements of DX_R in X_{DR}. This correspondence leads us to the following main result of this section.

THEOREM 4.6. *Let (D,R) be a skew-prime pair of nonsingular matrices. Let*

$$n := \sum_{i,j} \delta_{ij}, \tag{4.7}$$

where δ_{ij} is the degree of a greatest common divisor of the ith invariant factor of D and the jth invariant factor of R. Then $S(D,R)$ is in a bijective correspondence with K^n. Furthermore let $T := (T_1, T_2, \ldots, T_n)$ be an n-tuple of algebraically independent indeterminates. Then there exist polynomial matrices $\tilde{R}(z,T)$ and $\tilde{D}(z,T)$ such that the map

$$\tau: K^n \to S(D,R): (\zeta_1, \ldots, \zeta_n) \mapsto \left[\tilde{R}(z, \zeta_1, \ldots, \zeta_n), \tilde{D}(z, \zeta_1, \ldots, \zeta_n)\right]$$

is well defined and bijective.

Proof. Let $Q := DR$, $F := F_Q$, $V := X_Q$, and $V_1 := DX_R$. With this notation

$$\Gamma(D,R) = \{V_2 \subseteq V: FV_2 \subseteq V_2 \text{ and } V = V_1 \oplus V_2\}.$$

As (D, R) is skew-prime, $\Gamma(D, R)$ is nonempty. Let V_0 be any arbitrary but fixed element of $\Gamma(D, R)$. Let S_0 and S_1 be $p \times q$ and $p \times r$ polynomial matrices such that the columns of S_0 constitute a basis for V_0 and those of S_1 constitute a basis for V_1; clearly, $q := \dim V_0 = \deg\det D$ and $r := \dim V_1 = \deg\det R$. Let F_0 and F_1 respectively be the matrix representation of $F \mid V_0$ and $F \mid V_1$ with respect to the bases given by S_0 and S_1, i.e., F_0 is in $K^{q \times q}$ and F_1 is in $K^{r \times r}$ such that

$$F(S_0) = S_0 F_0, \qquad F(S_1) = S_1 F_1. \tag{4.8}$$

For any λ in $K^{r \times q}$, let

$$S_2(\lambda) := S_0 - S_1\lambda.$$

Since $V_0 \cap V_1 = \{0\}$, it follows from the definition of S_0 and S_1 that the columns of $S_2(\lambda)$ are K-linearly independent; hence the columns of $S_2(\lambda)$ constitute a basis for the q-dimensional K-linear space $S_2(\lambda)K^q$. Now, if x is a vector in $V_1 \cap S_2(\lambda)K^q$, then there exist g_1 in K^q and g_2 in K^r such that

$$S_2(\lambda)g_1 = S_0 g_1 - S_1 \lambda g_1 = S_1 g_2.$$

Again $V_0 \cap V_1 = \{0\}$ implies that $g_1 = 0$, and consequently $x = 0$. Therefore for each λ in $K^{r \times q}$

$$V = V_1 \oplus S_2(\lambda)K^q. \tag{4.9}$$

Let us now define the set

$$\Lambda := \{\lambda \text{ in } K^{r \times q} : \lambda F_0 = F_1 \lambda\}. \tag{4.10}$$

Now let λ be in Λ, and g be in K^q. Then we have

$$\begin{aligned} F(S_2(\lambda)g) &= F(S_0 g - S_1 \lambda g) \\ &= S_0 F_0 g - S_1 F_1 \lambda g = S_0 F_0 g - S_1 \lambda F_0 g \\ &= S_2(\lambda) F_0 g. \end{aligned}$$

Therefore $S_2(\lambda)K^q$ is an F-invariant subspace of V; now (4.10) implies that $S_2(\lambda)K^q$ belongs to $\Gamma(D, R)$ for each λ in Λ.

We can now define the map

$$\theta\colon \Lambda \to \Gamma(D, R) : \lambda \to S_2(\lambda)K^q.$$

We will show that θ is bijective. Let λ_1, λ_2 in Λ be such that $\theta(\lambda_1) = \theta(\lambda_2)$. Then there exists a nonsingular matrix T in $K^{q\times q}$ such that $S_2(\lambda_1) = S_2(\lambda_2)T$. Consequently

$$S_0 - S_1\lambda_1 = (S_0 - S_1\lambda_2)T.$$

which in turn implies that

$$S_0(I - T) = S_1(\lambda_1 - \lambda_2 T).$$

Since the columns of S_0 and S_1 respectively constitute bases for V_0 and V_1, it follows that

$$I = T \quad \text{and} \quad \lambda_1 = \lambda_2 T = \lambda_2.$$

Therefore θ is injective.

We will now prove that θ is also surjective. Let V_2 be any element of $\Gamma(D, R)$. Then there exists a λ in $K^{r\times q}$ such that the columns of the polynomial matrix $S_0 - S_1\lambda$ belong to V_2; in fact, the ith column of $S_1\lambda$ is the projection of the ith column of S_0 on V_1 along V_2. Hence the columns of $S_2 := S_0 - S_1\lambda$ are in V_2. Since the columns of S_2 are K-linearly independent and $\dim_K V_2 = q$, it follows that they constitute a basis for V_2. As V_2 is F-invariant, there exists an F_2 in $K^{q\times q}$ such that

$$F(S_2) = S_2F_2 = (S_0 - S_1\lambda)F_2.$$

On the other hand

$$F(S_2) = F(S_0 - S_1\lambda) = S_0F_0 - S_1F_1\lambda.$$

Since the columns of S_0 and S_1 respectively constitute bases for V_0 and V_1, it follows that

$$F_2 = F_0 \quad \text{and} \quad F_1\lambda = \lambda F_2 = \lambda F_0.$$

Hence λ belongs to Λ, and $\theta(\lambda) = S_2K^q = V_2$. Therefore θ is surjective.

Hence we have shown that θ is bijective. Note that Λ is the set of all solutions to the homogeneous *Lyapunov equation* $\lambda F_0 = F_1\lambda$. We can now use the classical results from the theory of the Lyapunov equation to exhibit a parametrization of $\Gamma(D, R)$.

Note that since $V_1 = DX_R$, it follows from Fuhrmann [9] that the invariant factors of F_1 are the same as those of R. Also, it is easy to see that the invariant factors of F_0 are the same as the invariant factors of D. Now, it follows from Gantmacher [13, p. 215] that Λ is an n-dimensional K-linear subspace of $K^{r\times q}$, where n is given by (4.7). Let $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be a basis for Λ. Then the map

$$(4.11)\quad \chi: K^n \to \Gamma(D, R): (\xi_1, \xi_2, \dots, \xi_n)' \to \left[S_0 - S_1\left(\sum_{i=1}^{n} \xi_i\lambda_i\right)\right]K^q$$

is well-defined and bijective. This completes the parametrization of $\Gamma(D, R)$. Proposition 4.4 now implies that K^n and $S(D, R)$ are in a bijective correspondence.

We now proceed to prove a sharper result: the existence of a polynomial parametrization of the space of equivalence classes of the skew-complements of (D, R).

Define the polynomial matrix

$$(4.12)\qquad \hat{S}(z, T) := S_0 - S_1\left(\sum_{i=1}^{n} T_i\lambda_i\right).$$

Let us now consider the rational matrix

$$Z(T) := Q^{-1}\hat{S}(z, T).$$

Clearly, $Z(T)$ is a strictly proper rational matrix in z with coefficients in the polynomial ring $K[T_1, T_2, \dots, T_n]$. Thus $Z(T)$ may be viewed as the *transfer matrix of a linear system over the polynomial ring* $K[T_1, T_2, \dots, T_n]$. (For basic definitions and results on linear systems over rings, see Sontag [26], Kamen [17], Sontag [27], and the references cited there.) Let $\hat{K}$ denote the algebraic closure of K. For any $\xi := (\xi_1, \xi_2, \dots, \xi_n)'$ in $\hat{K}^n$, let $Z(\xi)$ and $\hat{S}(\xi)$ respectively denote the transfer matrix and the polynomial matrix (in z over the field $\hat{K}$) obtained by substituting $T_i = \xi_i$ in $Z(T)$ and $\hat{S}(T)$. Then

$$Z(\xi) = Q^{-1}\hat{S}(\xi).$$

Let $\Sigma(Q, \hat{S}(\xi)) =: (\hat{F}, \hat{G}, \hat{H}, \hat{X})$ be the (observable) realization of $Z(\xi)$ over the field $\hat{K}$. It is easy to see that $\hat{S}(\xi)\hat{K}^q$ is an $\hat{F}$-invariant subspace of $\hat{X}$, and consequently $\hat{S}(\xi)\hat{K}^q$ is the reachable subspace of $\Sigma(Q, \hat{S}(\xi))$. Hence the McMillan degree of $Z(\xi)$ is q for all ξ in $\hat{K}^n$. It now follows from Sontag [26, Theorem 4.8] and Khargonekar [18, Theorem 5.9] that $Z(T)$ is a *split transfer matrix*. Furthermore Theorem 5.3 of Khargonekar [18] implies the existence of polynomial matrices (over the field K) $\tilde{D}(z,T)$, $\tilde{R}(z,T)$, $\tilde{S}(z,T)$, $Y_1(z,T)$, and $Y_2(z,T)$ such that

$$Q = \tilde{R}\tilde{D}, \qquad S = \tilde{R}\tilde{S}, \tag{4.13}$$

and

$$\tilde{D}Y_1 + \tilde{S}Y_2 = I. \tag{4.14}$$

For any ξ in K^n, let $\tilde{D}(\xi)$, $\tilde{R}(\xi)$, $\tilde{S}(\xi)$, $Y_1(\xi)$, and $Y_2(\xi)$ respectively denote the evaluation of the polynomial matrices $\tilde{D}$, $\tilde{R}$, $\tilde{S}$, Y_1, and Y_2 at $T = \xi$. Now (4.13) and (4.14) imply that $\tilde{R}(\xi)$ is the greatest common left divisor of Q and $\hat{S}(\xi)$. Hence, it follows that

$$\hat{S}(\xi)K^q = \tilde{R}(\xi)X_{\tilde{D}(\xi)},$$

for each ξ in K^n. Hence the bijection X can be rewritten as

$$\chi : K^n \to \Gamma(D, R) : \xi \to \tilde{R}(\xi)X_{\tilde{D}(\xi)}.$$

It is easy to verify that $\tau = \psi^{-1}\chi$, where ψ is as in the proof of Proposition 4.5. Finally, the map τ is bijective, as both χ and ψ are bijective. ∎

REMARK 4.15. The above theorem shows that the set $S(D, R)$ of equivalence classes of skew-complementary pairs is an n-parameter family, where n is given by (4.7). Furthermore, we have shown that there exists a family of representatives $(\tilde{R}, \tilde{D})$ of these equivalence classes which are *polynomially parametrized* by n algebraically independent parameters. Thus in principle, one can parametrize $S(D, R)$ by n independent parameters. However, the computational aspects of this parametrization are not clear.

Let us now use the results of Theorem 4.6 to find conditions under which the skew-complement pair is unique (up to unimodular equivalence). These conditions are described in the following easy

COROLLARY 4.16. *Let D and R be square nonsingular polynomial matrices. Then the set $S(D, R)$ (of equivalence classes of the skew-complement pairs of (D, R)) consists of a single element if and only if* $\det D$ *and* $\det R$ *are relatively prime.*

Proof. Let $\det D$ and $\det R$ be relatively prime. Then it is well known that there exist polynomial matrices $\tilde{R}$ and $\tilde{D}$ such that $DR = \tilde{R}\tilde{D}$ and $\det \tilde{R} = \det R$, $\det \tilde{D} = \det D$. Hence D and $\tilde{R}$ are left coprime and R and $\tilde{D}$ are right coprime. Thus, (D, R) is a skew-prime pair. Now we can apply the results of Theorem 4.6. Coprimeness of $\det D$ and $\det R$ implies that $n = 0$, and hence the set $S(D, R)$ consists of a single element.

Conversely, if $S(D, R)$ contains exactly one element, then (D, R) is a skew-prime pair and the number n of Theorem 4.6 must be 0. This implies that the elementary divisors of D and R are pairwise coprime. Thus $\det D$ and $\det R$ are relatively prime. ■

The result given in the above corollary was (partially) obtained by Wolovich [29, Theorem 6], who proved that if $\det D$ and $\det R$ are coprime then $S(D, R)$ contains exactly one element. Our result is somewhat stronger and shows that the coprimeness of $\det D$ and $\det R$ is also a necessary condition for the uniqueness of the skew-complement pair.

EXAMPLE 4.17. We now take a very simple example to illustrate the result of Theorem 4.6. Let us consider

$$D:=\begin{bmatrix} z-1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad R:=\begin{bmatrix} z+1 & 0 \\ 0 & z-1 \end{bmatrix}.$$

It is easy to see that (D, R) is a skew-prime pair. Also we can see that in this case $n = 1$. Hence the set $S(D, R)$ must be in bijective correspondence with K^1. In fact, some easy computations along the lines of the proof of Theorem 4.6 show that if we let

$$\tilde{R}(\xi):=\begin{bmatrix} z+1 & 0 \\ \xi & z-1 \end{bmatrix}, \qquad \tilde{D}(\xi):=\begin{bmatrix} z-1 & 0 \\ -\xi & 1 \end{bmatrix}$$

then the set $S(D, R)$ can be described as

$$S(D, R) = \{[\tilde{R}(\xi), \tilde{D}(\xi)] : \xi \in K\}.$$

Thus $S(D, R)$ is a one-parameter family as expected.

5. THE REGULATOR PROBLEM

Synthesis of feedback structures for output regulation and tracking with internal stability (ORIS) is one of the basic control theory problems. Thus, it is only natural that this problem has been considered by several investigators in the last few years. ORIS has been considered in various different frameworks: the state-space or geometric approach (for which see Wonham and Pearson [32], Wonham [31], and the references cited there), the polynomial matrix fraction representations (for which the reader is referred to Bengtsson [3], Cheng and Pearson [4], Wolovich and Ferreira [30], and the references given there), fractional representations over the rings of stable and proper stable transfers functions (as developed by Saeks and Murray [25], Pernebo [22, 23], Cheng and Pearson [5], and others). All these approaches lead to conditions for solvability of the output regulation problem.

It is clear that the problems considered in the various references given above are essentially the same. Therefore, it should be expected that the various conditions for the solvability of the ORIS are related, if not equivalent. The geometric framework leads to certain conditions in terms of the (F,G)-invariant and reachability subspaces, whereas the polynomial matrix framework involves the concept of skew-primeness. Using the results of Section 3, we will show that the conditions for solvability of ORIS given by Wonham and Pearson [32] and Wonham [31] are equivalent to those given by Wolovich and Ferreira [30]. This equivalence, though expected, is far from trivial.

Intuitively speaking, the problem of ORIS consists in obtaining a dynamic feedback scheme for a given (finite-dimensional) time-invariant, linear system to ensure the following desirable behavior:

(1) the closed-loop system consisting of the original system (plant) and the dynamic feedback scheme is internally stable, and

(2) the effect of the disturbances (that belong to a specified class of signals) on the output is asymptotically stable. Note that the problem of output tracking can be considered in this framework. (See Wonham [31, Chapter 6].)

Throughout this section we will work with the field of real numbers $\mathbb{R}$. Our results, being purely algebraic, are valid for discrete-time as well as continuous-time systems. For the sake of concreteness, we will work with discrete-time systems.

Let us consider a discrete-time finite-dimensional linear system $\Sigma = (F, G, H)$ with the dynamical equations

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t), \\ y(t) &= Hx(t), \end{aligned} \tag{5.1}$$

where $x(t)$ in $\mathbb{R}^n$, $u(t)$ in $\mathbb{R}^m$, and $y(t)$ in $\mathbb{R}^p$ respectively denote the state, the input, and the output of the system Σ at the time t. We assume that the disturbance signals are the outputs of a linear discrete-time system, usually called the *exogenous system*. Then we can model the effect of the disturbances on the plant by the following equations:

$$\begin{aligned} x(t+1) &= Fx(t)+F_c x_d(t)+Gu(t), \\ x_d(t+1) &= F_d x_d(t), \qquad (5.2) \\ y(t) &= Hx(t)+H_d x_d(t). \end{aligned}$$

Here $x_d(t)$ in $\mathbb{R}^{n_d}$ represents the state of disturbance. Note that the disturbance affects the state of the plant as well as the output of the plant. Thus the overall model of the plant and the disturbances is completely described by the system $\hat{\Sigma}:=(\hat{F},\hat{G},\hat{H})$, where

$$(5.3) \qquad \hat{F}:=\begin{bmatrix} F & F_c \\ 0 & F_d \end{bmatrix}, \qquad \hat{G}:=\begin{bmatrix} G \\ 0 \end{bmatrix}, \qquad \hat{H}:=\begin{bmatrix} H & H_d \end{bmatrix}.$$

We can now formulate the problem of *output regulation with internal stability* as follows:

Given the linear system $\hat{\Sigma}=(\hat{F},\hat{G},\hat{H})$, *find an* $m\times p$ *transfer matrix* Z_1 *and an* $m\times m$ *strictly proper transfer matrix* Z_2, *such that the feedback law*

$$u = Z_1 y + Z_2 u + v,$$

where v *is a possible external input*, *results in*

(i) *the internal stability of the closed-loop system consisting of the plant* $\Sigma=(F,G,H)$ *and the canonical realization* Σ_c *of the dynamic compensator* $Z_c=[Z_1\ \ Z_2]$, *and*

(ii) *the asymptotic convergence to zero of the output of the closed-loop system for all initial states of the system* $\hat{\Sigma}$.

REMARK 5.4. The feedback scheme considered above subsumes the dynamic output feedback scheme of Wolovich and Ferreira [30] and the observer and (dynamic) state feedback scheme of Wonham [31, Chapter 7] as special cases. In fact, it is not difficult to verify that the problem of ORIS as formulated above can be solved if and only if it can be solved by dynamic output feedback (i.e., $Z_2 = 0$), if and only if it can be solved by an observer and (dynamic) state feedback. This observation allows us to use the results of Wonham [31] and Wolovich and Ferreira [30] for the problem formulated above.

Since we are interested only in the asymptotic behavior of the closed loop system, there is no loss of generality in assuming that $\chi(F_d)$, *the characteristic polynomial of* F_d, *is completely unstable.* In order to make our presentation reasonably simple we make two further assumptions. We assume that *the plant* Σ *is canonical and the overall system* $\hat{\Sigma}$ *is observable.* It is well known that these conditions are not very restrictive, as the general problem can be reduced to a problem satisfying these assumptions by using Kalman's canonical decomposition. These assumptions turn out to be very useful in establishing the link between the geometric and the polynomial matrix approaches to the problem of ORIS.

The first step in relating the results of Wonham [31] and Wolovich and Ferreira [30] is to obtain polynomial fraction representations "corresponding" to the system $\hat{\Sigma}$. This is achieved in the following

PROPOSITION 5.5. *Let* $\hat{\Sigma} = (\hat{F}, \hat{G}, \hat{H})$ *be an observable system, where* $\hat{F}, \hat{G}, \hat{H}$ *are as in* (5.3). *Then there exist polynomial matrices* Q, D, R *such that*

$$Q^{-1}R = (DQ)^{-1}DR = \hat{H}(zI - \hat{F})^{-1}\hat{G} = H(zI - F)^{-1}G,$$

the system $\Sigma(DQ, DR)$ *is isomorphic with* $\hat{\Sigma}$, *and the system* $\Sigma(Q, R)$ *is isomorphic with* $\Sigma = (F, G, H)$. *If* D_1, Q_1, R_1 *are polynomial matrices such that* $\Sigma(D_1Q_1, D_1R_1)$ *is isomorphic with* $\hat{\Sigma}$ *and* $\Sigma(Q_1, R_1)$ *is isomorphic with* Σ, *then there exist unimodular polynomial matrices* M *and* N *such that*

$$Q_1 = NQ, \qquad R_1 = NR, \qquad D_1 = MDN^{-1}.$$

The pair (F, G) *is reachable if and only if* Q *and* R *are left coprime. Further,*

$$\det D = \det(zI - F_d).$$

Proof. Let Q, S be left coprime polynomial matrices such that

$$H(zI - F)^{-1} = Q^{-1}S.$$

We then have

$$\hat{H}(zI-\hat{F})^{-1}=Q^{-1}\left[S:(-SF_c+QH_d)(zI-F_d)^{-1}\right].$$

Let D, S_d be left coprime polynomial matrices such that

$$(-SF_c+QH_d)(zI-F_d)^{-1}=D^{-1}S_d.$$

Consequently

$$\hat{H}(zI-\hat{F})^{-1}=(DQ)^{-1}(DS:S_d).$$

Using left coprimeness of D, S_d and Q, S, it is easy to verify (via the appropriate Bezout conditions) that DQ and $(DS:S_d)$ are left coprime polynomial matrices. Now it follows from Theorem 2.8 of Emre and Hautus [8] that with $R:=SG$, the system $\Sigma(DQ, DR)$ is isomorphic with $\hat{\Sigma}$ and the system $\Sigma(Q, R)$ is isomorphic with Σ. Theorem 2.4 now implies that (F, G) is reachable if and only if Q, R are left coprime.

If polynomial matrices D_1, Q_1, R_1 satisfy the hypothesis of the proposition, then it follows from [10] that there exist unimodular matrices M, N such that $Q_1=NQ$, $R_1=NR$, and $D_1Q_1=MDQ$.

Let us now prove that $\det D=\det(zI-F_d)$. Since $H(zI-F)^{-1}=Q^{-1}S$ are coprime factorizations of the same rational matrix, it follows that $\deg Q=\det(zI-F)$. Similarly,

$$\det(DQ)=\det(zI-\hat{F})=\det(zI-F)\det(zI-F_d).$$

Hence, $\det D=\det(zI-F_d)$. ■

REMARK 5.6. The above proposition shows how to "translate" the state-space data $(\hat{F},\hat{G},\hat{H})$ into polynomial matrices. The factorization $Q^{-1}R$ corresponds to the plant (F, G, H). The polynomial matrix D corresponds to the disturbance signals. Since $\chi(F_d)$ is assumed to be completely unstable, it follows that $\det D$ is completely unstable. Further, we also assumed that Σ is reachable. Therefore we may assume that Q, R are left coprime. With these observations, it is easy to see that the problem of ORIS above corresponds exactly to the setup of Wolovich and Ferreira [30] with the simplifying assumption that the plant is canonical and the plant outputs are measured directly.

Proposition 5.5 implies that instead of working with the overall system $\hat{\Sigma}$, we may as well work with the isomorphic system $\Sigma(DQ, DR)$. From now on we will fix polynomial matrices Q, D, R such that $\Sigma(DQ, DR)$ is isomorphic with the observable system $\hat{\Sigma}$, and $\Sigma(Q, R)$ is isomorphic with the canonical system Σ.

In view of the above discussion, we now reformulate the problem of output regulation with internal stability in terms of the polynomial matrices D, Q, R.

DEFINITION 5.7 (ORIS). Let D, Q, R be polynomial matrices such that DQ is nonsingular, D and R are left coprime, and $\det D$ is completely unstable. Let $y=(DQ)^{-1}DRu$ be the input-output description of the (extended) system in a left-matrix-fraction-description form. Find a causal transfer matrix Z_1 and a strictly causal transfer matrix Z_2 such that the feedback law

$$u = Z_1 y + Z_2 u + w,$$

where w is a possible external input, results in

(i) the internal stability of the closed-loop system consisting of $\Sigma(DQ, DR) =:(\tilde{F}, \tilde{G}, \tilde{H})$ and the canonical realization Σ_c of $Z_c := [Z_1 \ \ Z_2]$, and

(ii) the asymptotic convergence to zero of the output of the closed-loop system for all initial states of $\Sigma(DQ, DR)$ and Σ_c

The second step in relating the geometric approach to the polynomial matrix approach is to express various invariant subspaces involved in the setup of Wonham [31] in terms of polynomial matrices Q, D, R. In particular, the intersection of the largest $(\tilde{F}, \tilde{G})$-invariant subspace in $\ker \tilde{H}$ and the reachable subspace of $\Sigma(DQ, DR)$ plays an important role in the geometric approach to ORIS. This subspace is characterized in terms of polynomial matrices D, Q, R in the following

LEMMA 5.8. *The intersection of the largest $(\tilde{F}, \tilde{G})$-invariant subspace in* $\ker \tilde{H}$ *and the reachable subspace of* $\Sigma(DQ, DR)$ *is given by* DX_R.

Proof. By Theorem 2.7 the largest $(\tilde{F}, \tilde{G})$-invariant subspace in $\ker \tilde{H}$ is given by X_{DR}. Since Q, R are left coprime, it follows from [9] that the reachable subspace of $\Sigma(DQ, DR)$ is DX_Q. Let x be in $X_{DR} \cap DX_Q$. Then there exists a strictly proper y such that $x = DRy$. Since x is also in DX_Q, $D^{-1}x = Ry$ is polynomial, and hence belongs to X_R. Therefore

$$X_{DR} \cap DX_Q \subseteq DX_R.$$

Conversely, let x be in DX_R. Since $Q^{-1}R$ is a strictly proper transfer matrix, $X_R \subseteq X_Q$. Therefore $DX_R \subseteq DX_Q$. It is trivial to verify that $DX_R \subseteq X_{DR}$. Therefore

$$DX_Q \cap X_{DR} = DX_R. \qquad \blacksquare$$

We can now state the results of Wonham [31, Chapter 7] on the solvability of ORIS in terms of polynomial matrices Q, D, R.

THEOREM 5.9 (Wonham [31, Corollary 7.3]). *The problem of ORIS of* (5.7) *has a solution if and only if*

(5.10i) $X_{DQ} = X_{DR} + DX_Q$,

(5.10ii) DX_R/R_m *decomposes* X_{DR}/R_m *relative to the map induced on* X_{DR}/R_m *by* $\tilde{F}_L := \tilde{F} + \tilde{G}L$, *for any* L *in* $\mathbf{L}(X_{DR})$, *where* R_m *represents the largest* $(\tilde{F}, \tilde{G})$*-reachability subspace contained in* $\ker \tilde{H}$.

In view of Proposition 5.5 and Remark 5.6, the results of Wolovich and Ferreira [30] imply the following

THEOREM 5.11 (Wolovich and Ferreira [30]). *The problem of ORIS as in* (5.7) *has a solution if and only if the ordered pair* (D, R) *is skew-prime.*

We are now ready to state the main result of this section, which shows the relation between Theorems 5.9 and 5.11.

THEOREM 5.12. *The ordered pair* (D, R) *is skew-prime if and only if*

(5.10i) $X_{DQ} = DX_Q + X_{DR}$,

(5.10ii) DX_R/R_m *decomposes* X_{DR}/R_m *relative to the map induced on* X_{DR}/R_m *by* $\tilde{F}_L := \tilde{F} + \tilde{G}L$, *for any* L *in* $\mathbf{L}(X_{DR})$.

Proof. Suppose that the ordered pair (D, R) is skew-prime. By Theorem 3.3 there exists a Y-shift-module structure on X_{DR} such that

$$X_{DR} = DX_R \underset{\mathbb{R}[z]}{\oplus} V$$

for some $V \subseteq X_{DR}$. Note that by Theorem 2.7, there exists an L in $\mathbf{L}(X_{DR})$ such that for all x in X_{DR},

$$z \cdot x = \tilde{F}_L x. \tag{5.13}$$

Now let $W: = V + R_m$. As R_m is an $\mathbb{R}[z]$-submodule of X_{DR} and is contained in DX_R, it follows that

$$X_{DR}/R_m = DX_R/R_m \underset{R[z]}{\oplus} W/R_m.$$

The above $\mathbb{R}[z]$-direct-sum decomposition and (5.13) imply that DX_R/R_m decomposes X_{DR}/R_m relative to the linear map induced by $\tilde{F}_L$. Finally, by Theorem 2.8, the linear map induced by $\tilde{F}_L$ is the same for all L in $\mathbb{L}(X_{DR})$. Hence the second condition in (5.10) holds. As (D, R) is skew-prime, Theorem 3.4 implies that

$$\dim X_{DR} = \dim X_D + \dim X_R.$$

Now $X_{DR} \cap DX_Q = DX_R$. It is clear that

$$X_{DR} + DX_Q \subseteq X_{DQ}.$$

However,

$$\dim(X_{DR} + DX_Q) = \dim X_{DR} + \dim DX_Q - \dim DX_R.$$

As D is nonsingular, it follows that

$$\dim(X_{DR} + DX_Q) = \dim X_D + \dim X_Q = \dim X_{DQ}.$$

Hence

$$X_{DR} + DX_Q = X_{DQ}.$$

Thus, if (D, R) is skew-prime, then (5.10i, ii) hold.

Now suppose (5.10i, ii) hold. Since $X_{DR} \cap DX_Q = DX_R$, (5.10i) implies that

$$\dim X_{DQ} = \dim X_{DR} + \dim DX_Q - \dim DX_R.$$

As the polynomial matrices D and Q are nonsingular, we have $\dim X_{DQ} = \dim X_D + \dim X_Q$ and hence

$$\dim X_{DR} = \dim X_D + \dim X_R. \tag{5.14}$$

Let ψ denote the product of the invariant factors of the polynomial matrix DR. Now choose an L in $\mathbb{L}(X_{DR})$ such that the characteristic polynomial of the linear map $\tilde{F} + \tilde{G}L =: \tilde{F}_L$ restricted to R_m is coprime with ψ. Such an L exists because $\mathbb{R}$ is an infinite field and coefficient assignability holds for the space R_m. (For details see Wonham [31, Chapter 5], Emre and Hautus [8, Section 6]). Now (5.10ii) implies that there exists an $\tilde{F}_L$-invariant subspace $W \supseteq R_m$ such that

$$X_{DR}/R_m = DX_R/R_m \oplus W/R_m.$$

By Theorem 2.8 the nontrivial invariant factors of X_{DR}/R_m are exactly the nontrivial invariant factors of the polynomial matrix DR. Hence the characteristic polynomial of the linear map induced by $\tilde{F}_L$ on X_{DR}/R_m is ψ. Since $\tilde{F}_L R_m \subseteq R_m$ and the characteristic polynomial of $\tilde{F}_L$ restricted to R_m is coprime with ψ, it follows from elementary results in linear algebra that there exists an $\tilde{F}_L$-invariant subspace $V_1 \subseteq X_{DR}$ such that

$$X_{DR} = R_m \oplus V_1.$$

Let $V := W \cap V_1$. Therefore V is also $\tilde{F}_L$-invariant. It is now easy to verify (using the fact that $DX_R \supseteq R_m$) that

$$X_{DR} = DX_R \oplus V,$$

where both DX_R and V are $\tilde{F}_L$-invariant. Now consider the Y-shift-module structure on X_{DR} corresponding to the linear map $\tilde{F}_L$. It follows that

$$X_{DR} = DX_R \underset{\mathbb{R}[z]}{\oplus} V. \tag{5.15}$$

Now (5.14) and (5.15) together with Theorem 3.3 imply that the ordered pair (D, R) is skew-prime ∎

Theorems 5.9, 5.11, and 5.12 show that the conditions for the solvability of the problem of ORIS via the geometric approach and via the polynomial matrix approach are equivalent. Even though this result is not unexpected, the proof of Theorem 5.12 involves nontrivial applications of the techniques developed in Emre and Hautus [8] and Khargonekar and Emre [19]. Results of these papers, together with those of this paper, clearly indicate that there is a close connection between the results obtained via the state-space or geometric approach and the polynomial matrix approach.

REMARK 5.16. We have not paid any attention to the actual computations of the feedback compensators. It seems quite likely that the feedback structures obtained by Wonham [31] and Wolovich and Ferreira [30] are also closely related. This is evident in view of the fact that both of these techniques lead to an internal model principle. We leave a thorough investigation of these ideas for further work.

REMARK 5.17. Note that in the proof of Theorem 5.12 we explicitly used the fact that the real field is infinite. Theorem 5.12 makes sense for any field, even though the regulator problem may not. As may be expected, *Theorem* 5.12 *is true for all fields*. Clearly, our proof is valid for any infinite field. In the general case the proof becomes considerably more involved. Note that R_m is the largest reachability space in X_{DR}. Using this fact, one can prove that *there exists an* L *in* $\mathrm{L}(X_{DR})$ *such that* R_m *admits an* $\tilde{F}_L$*-invariant complement*. The rest of the proof remains the same. We do not go into these details, since we are mainly interested in the real field in this section, and since the proof in the general case does not have any unexpected features.

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